NET & SET Economics Mathematical Economics

Prepared by Dr.Dubba Ranjith

1. Sets, Functions and Continuity; Sequences and Series

Sets

- A set is a well-defined collection of objects. Notation: $A = \{1, 2, 3\}$; $x \in A$.
- Types: finite, infinite, null ($\emptyset$), singleton, universal (U), subset ($A \subseteq B$), proper subset, power set (2^n subsets for n elements), equal sets.
- **Operations:** union ($A \cup B$), intersection ($A \cap B$), difference ($A - B$), complement (A').
- **Laws:** commutative, associative, distributive; **De Morgan's laws:** $(A \cup B)' = A' \cap B'$ and $(A \cap B)' = A' \cup B'$.
- $n(A \cup B) = n(A) + n(B) - n(A \cap B)$.
- **Cartesian product** $A \times B = \{(a, b): a \in A, b \in B\}$; relation = subset of $A \times B$.
- **Convex set:** the line segment joining any two points lies wholly in the set — central to optimisation and general equilibrium (budget sets, production sets).

Functions

- A function $y = f(x)$ assigns to each element of the domain exactly one element in the range (one-to-one or many-to-one, never one-to-many).
- Types: constant, linear ($y = a + bx$), quadratic, polynomial, rational (rectangular hyperbola $y = a/x$ — AFC curve), exponential ($y = a^{bx}$), logarithmic, power.
- **Homogeneous function of degree n:** $f(\lambda x, \lambda y) = \lambda^n f(x, y)$. **Homothetic function:** monotonic transformation of a homogeneous function.
- **Concave and convex functions:** f is concave if $f[\theta x_1 + (1-\theta)x_2] \geq \theta f(x_1) + (1-\theta)f(x_2)$. Quasi-concave functions have convex upper contour sets (well-behaved utility functions).
- Inverse function, composite function $f(g(x))$.

Limits and continuity

- $\lim(x \to a) f(x) = L$ if $f(x)$ approaches L as x approaches a; left-hand limit = right-hand limit.
- **Continuity at x = a:** $f(a)$ is defined, the limit exists, and $\lim f(x) = f(a)$.
- A differentiable function is continuous, but a continuous function need not be differentiable (e.g., $|x|$ at 0, kinked demand curve).
- **Weierstrass (extreme value) theorem:** a continuous function on a closed, bounded set attains a maximum and minimum.
- **Intermediate value theorem**; Brouwer's fixed-point theorem (used in proving existence of general equilibrium).

Sequences and series

- **Arithmetic progression:** a, a + d, …; nth term = $a + (n-1)d$; sum $S_n = n/2\ [2a + (n-1)d]$.

- **Geometric progression:** a, ar, ar², …; nth term = ar^{n-1}; $S_n = a(1 - r^n)/(1 - r)$; infinite sum = $a/(1 - r)$ for $|r| < 1$ — basis of the multiplier ($1 + b + b^2 + \ldots = 1/(1 - b)$).
- Convergent vs divergent sequences.
- **Compound interest:** $A = P(1 + r)^n$; continuous compounding $A = Pe^{rt}$; present value $PV = A/(1 + r)^n$.
- $e = \lim(n \to \infty) (1 + 1/n)^n \approx 2.718$.

Exam pointers: De Morgan's laws; convex sets; homogeneous functions; GP sum and multiplier; continuity vs differentiability.

2. Differential Calculus and its Applications

Rules of differentiation

Rule	Result
Power	$d(x^n)/dx = nx^{n-1}$
Sum	$d(u \pm v) = du \pm dv$
Product	$d(uv)/dx = u \cdot v' + v \cdot u'$
Quotient	$d(u/v)/dx = (v \cdot u' - u \cdot v')/v^2$
Chain	$dy/dx = (dy/du)(du/dx)$
Exponential	$d(e^x)/dx = e^x$; $d(a^x)/dx = a^x \ln a$
Logarithmic	$d(\ln x)/dx = 1/x$

Partial and total derivatives

- **Partial derivative** $\partial f/\partial x$ holds other variables constant (marginal product, marginal utility).
- **Total differential:** $dz = f_x dx + f_y\, dy$.
- **Implicit function rule:** $dy/dx = -f_x/f_y$ (slope of indifference curve = −MUx/MUy; slope of isoquant).
- **Young's theorem:** cross partials are equal: $f_{x\gamma} = f_{\gamma x}$.
- **Euler's theorem:** for f homogeneous of degree n, $x \cdot f_x + y \cdot f_y = n \cdot f$.

Economic applications

- **Marginal concepts:** MC = dTC/dQ; MR = dTR/dQ; MU, MP, MPC = dC/dY.
- **Elasticity:** $e = (dQ/dP)(P/Q) = d(\ln Q)/d(\ln P)$. For $Q = aP^{-b}$ elasticity is constant = b.
- $MR = AR(1 - 1/e)$; AR, MR and elasticity relationship.
- Relationship between average and marginal: $d(AC)/dQ = (MC - AC)/Q \rightarrow$ MC cuts AC at its minimum.
- **Growth rates:** if y = uv, growth of y = growth of u + growth of v; $d(\ln y)/dt$ = instantaneous growth rate.
- Comparative statics of market models (effect of tax on equilibrium price).

Integral calculus (applications)

- $\int x^n dx = x^{n+1}/(n+1) + C$; $\int (1/x) dx = \ln x + C$; $\int e^x dx = e^x + C$.
- $TC = \int MC\, dQ + FC$; $TR = \int MR\, dQ$.

- **Consumer's surplus** = $\int_0^{Q_0} D(Q)dQ - P_0Q_0$; **producer's surplus** = $P_0Q_0 - \int_0^{Q_0} S(Q)dQ$.
- Present value of a continuous income stream = $\int R(t)e^{-rt}\,dt$; capital formation from investment flow $K(t) = \int I(t)dt$.

Exam pointers: product/quotient/chain rules; elasticity via derivatives; Euler's theorem; consumer surplus by integration; MR = AR(1 − 1/e).

3. Linear Algebra: Matrices and Vector Spaces

Matrices

- Types: row, column, square, diagonal, scalar, identity (I), null, symmetric ($A = A'$), skew-symmetric, triangular, idempotent ($A^2 = A$), orthogonal ($A'A = I$), singular ($|A| = 0$).
- Operations: addition (same order), scalar multiplication, matrix multiplication (columns of A = rows of B; generally $AB \neq BA$).
- **Transpose:** $(AB)' = B'A'$.
- **Determinant:** expansion by cofactors; $|AB| = |A||B|$; interchanging two rows changes sign; a row of zeros or two identical rows → determinant = 0.
- **Inverse:** $A^{-1} = \text{adj}(A)/|A|$, exists only if $|A| \neq 0$ (non-singular); $(AB)^{-1} = B^{-1}A^{-1}$.
- **Rank:** maximum number of linearly independent rows/columns; order of largest non-zero minor.
- **Trace:** sum of diagonal elements = sum of eigenvalues.

Solving linear equations AX = B

- **Cramer's rule:** $x_i = |A_i|/|A|$.
- Matrix inversion: $X = A^{-1}B$.
- Gaussian elimination.
- Consistency: system is consistent if rank(A) = rank(A|B); unique solution if rank = number of unknowns.

Vector spaces

- Vectors, linear combinations; **linear dependence/independence**; basis and dimension; span.
- Inner (dot) product; orthogonal vectors (inner product = 0); norm.
- **Eigenvalues and eigenvectors:** $|A - \lambda I| = 0$ (characteristic equation). Product of eigenvalues = $|A|$.

Quadratic forms and definiteness

- $Q = X'AX$. Positive definite if all leading principal minors > 0; negative definite if they alternate in sign starting negative ($|A_1| < 0$, $|A_2| > 0$, …).
- Used for second-order conditions of optimisation (Hessian).
- **Jacobian determinant |J|:** tests functional dependence; $|J| \neq 0$ → functions are independent.
- **Hessian |H|** (second-order partials) and **bordered Hessian** (constrained optimisation).

Exam pointers: conditions for inverse; Cramer's rule; rank and consistency; definiteness from principal minors; Jacobian vs Hessian.

4. Static Optimisation Problems and their Applications

Unconstrained optimisation — one variable

- First-order condition (necessary): $f'(x) = 0$.
- Second-order condition: $f''(x) < 0 \rightarrow$ maximum; $f''(x) > 0 \rightarrow$ minimum; $f'' = 0 \rightarrow$ inconclusive (point of inflexion possible).
- Application: profit maximisation $\pi = TR - TC \rightarrow MR = MC$ and slope of MC > slope of MR.

Unconstrained — several variables

- FOC: all first partials = 0.
- SOC (two variables): maximum if $f_{xx} < 0$, f_yy < 0 and $f_{xx}f_yy - (f_{x\gamma})^2 > 0$; minimum if $f_{xx} > 0$, f_yy > 0 and the same determinant > 0. In general, Hessian negative definite → maximum; positive definite → minimum.
- Application: multi-product firm, price discrimination across markets.

Constrained optimisation — Lagrange multiplier method

Maximise U(x, y) subject to $P_x x + P_y\, y = M$: $L = U(x, y) + \lambda(M - P_x x - P_y\, y)$

- FOCs: $U_x = \lambda P_x$, $U_y = \lambda P_y$, budget constraint → $MUx/Px = MUy/Py = \lambda$.
- **λ = marginal utility of money** (shadow price of the constraint — rate of change of the optimal objective with respect to the constraint).
- SOC: **bordered Hessian** $|\bar{H}| > 0$ for maximum (two variables, one constraint); < 0 for minimum.
- Applications: utility maximisation, cost minimisation (least-cost combination), output maximisation, derivation of demand functions.

Inequality constraints

- **Kuhn–Tucker conditions** (non-linear programming): complementary slackness — either the constraint binds or its multiplier is zero.
- **Envelope theorem:** effect of a parameter change on the optimal value equals its direct partial effect on the Lagrangian (Roy's identity, Shephard's lemma, Hotelling's lemma follow from it).

Comparative statics

- Effect of parameter changes on equilibrium values using implicit function theorem and Cramer's rule.

Exam pointers: SOCs for max/min; interpretation of λ; bordered Hessian sign; Kuhn–Tucker complementary slackness; envelope theorem.

5. Input–Output Model and Linear Programming

Leontief Input–Output Analysis

- Developed by **Wassily Leontief** (1936, 1941; Nobel 1973). Shows inter-industry flows of goods.

- **Transaction table:** rows show distribution of each sector's output (intermediate use + final demand); columns show inputs used by each sector (intermediate inputs + primary inputs like labour).
- **Technical coefficient** $a_{ij} = x_{ij}/X_j$ (input of i per unit of output of j).

$$X = AX + D \rightarrow X = (I - A)^{-1} D$$

- $(I - A)$ is the **Leontief matrix**; $(I - A)^{-1}$ is the **Leontief inverse** (matrix multiplier) — gives total (direct + indirect) requirements.
- **Hawkins–Simon conditions:** for a viable (non-negative) solution, all principal minors of $(I - A)$ must be positive (each sector must produce more than it uses of its own output; column sums of $A < 1$).
- Assumptions: constant returns, fixed coefficients (no substitution), each industry produces one product, no joint production.
- **Open model** (final demand exogenous) vs **closed model** (households treated as an industry).
- Uses: development planning (Mahalanobis model in India), forecasting, impact analysis, employment and pollution analysis.

Linear Programming

- Optimises a linear objective function subject to linear inequality constraints and non-negativity. Developed by **Kantorovich** (1939) and **George Dantzig** (simplex method, 1947); Kantorovich and Koopmans won the Nobel in 1975.
- Components: objective function, decision variables, constraints, non-negativity conditions.
- **Feasible region** is a convex set; the optimum lies at a **corner (extreme) point** (fundamental theorem of LP).
- Solution methods: **graphical** (two variables), **simplex method**, slack/surplus/artificial variables, Big-M method.
- Special cases: unbounded solution, infeasible solution, multiple optima, degeneracy.

Duality

- Every LP (primal) has a dual: a primal maximisation becomes a dual minimisation; the constraint constants of one become the objective coefficients of the other.
- **Duality theorem:** optimal values of primal and dual objectives are equal.
- Dual variables = **shadow prices** (imputed value of an extra unit of each resource).
- **Complementary slackness:** if a resource is not fully used, its shadow price is zero.
- Related: transportation and assignment problems; **game theory** (two-person zero-sum games can be solved as LP).

Exam pointers: Leontief inverse; Hawkins–Simon conditions; corner-point theorem; duality and shadow prices; Dantzig's simplex.

6. Difference and Differential Equations with Applications

Dynamic economics studies the time path of variables. Continuous time uses differential equations; discrete (period) time uses difference equations.

First-order differential equations

$$dy/dt + ay = b \rightarrow y(t) = [y(0) - b/a]e^{-at} + b/a$$

- b/a = particular integral (intertemporal equilibrium); the first term = complementary function (deviation).
- **Stability:** converges to equilibrium if $a > 0$ (exponent negative).
- **Application — Walrasian price adjustment:** $dP/dt = \alpha(Qd - Qs)$; stable if demand slopes down and supply slopes up.
- Growth models: Harrod–Domar ($dK/dt = sY$), Solow ($dk/dt = sf(k) - (n + \delta)k$); Domar's debt burden model.

Second-order differential equations

- $y'' + a_1y' + a_2y = b$; characteristic roots $r_1, r_2 = [-a_1 \pm \sqrt{(a_1^2 - 4a_2)}]/2$.
- Distinct real roots, repeated roots, or complex roots (oscillation). Stable if real parts of roots are negative.
- Application: price dynamics with expectations, inventory models.

First-order difference equations

$$y_{t+1} + ay_t = c \rightarrow y_t = y_0 - c/(1+a)^t + c/(1+a)$$

- Let $b = -a$. Time path: $|b| < 1 \rightarrow$ convergent; $|b| > 1 \rightarrow$ divergent; $b < 0 \rightarrow$ oscillating; $b > 0 \rightarrow$ non-oscillating.
- **Cobweb model (Ezekiel, 1938):** $Qd_t = \alpha - \beta P_t$; $Qs_t = -\gamma + \delta P_{t-1}$ — supply responds to last period's price. Stable (convergent) if slope of supply < slope of demand in absolute terms ($\delta < \beta$); explosive if $\delta > \beta$; continuous (perpetual) oscillation if $\delta = \beta$.
- Harrod model in discrete form; Keynesian dynamic multiplier $Y_t = C_0 + bY_{t-1} + I_0$.

Second-order difference equations

- $y_{t+2} + a_1y_{t+1} + a_2y_t = c$; characteristic equation $b^2 + a_1b + a_2 = 0$.
- **Samuelson's multiplier–accelerator model:** $Y_t = G_0 + c(1 + v)Y_{t-1} - cvY_{t-2}$. Depending on c (MPC) and v (accelerator), time path is monotonic or oscillatory, damped or explosive ($cv < 1 \rightarrow$ damped).
- Hicks's trade cycle; Metzler's inventory cycle.

Exam pointers: stability conditions for difference and differential equations; cobweb stability condition; Samuelson interaction ($cv < 1$ damped).